HOMOLOGY STABILITY FOR UNITARY GROUPS II

B. MIRZAII

ABSTRACT. In this note the homology stability problem for hyperbolic unitary groups over a local ring with an infinite residue field is studied.

1. INTRODUCTION

In this note we continue the study of the homology stability problem for hyperbolic unitary groups, started in [7]. In [7] a general statement about homology stability for these groups was established. It was believed that similar to the general linear group case, for hyperbolic unitary groups over an infinite field, one can have a better range of homology stability. But no proof of this exists in the literature. Our main goal is the study of this problem.

Our main theorem asserts that for a local ring with an infinite residue field, the natural map $H_l(\text{inc}) : H_l(G_n, \mathbb{Z}) \to H_l(G_{n+1}, \mathbb{Z})$ is surjective for $n \geq l+1$ and is injective for $n \geq l+2$, where $G_n := U_{2n}^{\epsilon}(R, \Lambda)$ always with the underlying hyperbolic form. With a field k as the coefficient group we get even better result; $H_l(\text{inc}) : H_l(G_n, k) \to H_l(G_{n+1}, k)$ is surjective for $n \geq l$ and is injective for $n \geq l+1$. In fact the first result follows from the second one.

To get the second result, we will introduce some posets similar to one introduced and studied in [7]. In section 1 we prove that they are highly acyclic. Applying this we will come up with a spectral sequence, Theorem 3.5, which is the main purpose of section 3. The main difficulty is to analyze this spectral sequence which is done in section 4. The stability theorem will be a result of this analysis. An application of the stability theorem is given in this section. In section 5, we will discuss the homology stability problem in the case of a finite field.

I would like to thank W. van der Kallen for his useful comments that made some of the original proofs shorter and for his help in some of the proofs.

Here we will establish some notations. By a ring R, we will always mean a local ring with an infinite residue field unless it is mentioned. The ring R has an involution (which may be the identity) and we set $R_1 := \{r \in R : \overline{r} = r\}$. This is also a local ring with an infinite residue field. For the definition of the concepts that we use such as a bilinear form h, a hyperbolic unitary group and its elementary group, an isotropic element or set, the unimodular poset $\mathcal{U}(R^n)$, the isotropic unimodular poset $\mathcal{IU}(R^{2n})$ etc, we refer to [7,

sections 6 and 7]. We denote a hyperbolic unitary group $U_{2n}^{\epsilon}(R,\Lambda)$ and its elementary group by G_n and E_n respectively. By convention G_0 will be the trivial group. The embeddings $G_n \to G_{n+1}$ and $E_n \to E_{n+1}$ are given by $A \mapsto \mathrm{diag}(I_2, A) = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}$. For a group G, by $H_i(G)$ we mean $H_i(G,k)$, where k is a field with trivial G-action. By k as the coefficient group of the homology functor, we always mean a field. In some cases, which will be mentioned, it has to be prime field.

2. Isotropic unimodular posets

The main statement of this section, Theorem 2.5, is rather well known (see [9]). We give the details of the proof to make sure that everything is working for our case, Theorem 2.7. For an alternative proof in the case of a field different from $\mathbb{F}_2$ see Remark 1 and Theorem 5.1.

Definition 2.1. Let $S = \{v_1, \dots, v_k\}$ and $T = \{w_1, \dots, w_{k'}\}$ be basis of two isotropic free summands of R^{2n}. We say that T is in general position with S, if $k \le k'$ and the $k' \times k$-matrix $(h(w_i, v_j))$ has a left inverse.

Proposition 2.2. *Let $n \ge 2$ and assume T_i, $1 \le i \le l$, are finitely many finite subsets of R^{2n} such that each T_i is a basis of a free isotropic summand of R^{2n} with k elements, where $k \le n-1$. Then there is a basis, $T = \{w_1, \dots, w_n\}$, of a free isotropic summand of R^{2n} such that T is in general position with all T_i. Moreover $\dim(W \cap V_i^{\perp}) = n-k$, where $W = \langle T \rangle$ and $V_i = \langle T_i \rangle$.*

Proof. The proof of the first part is by induction on l. Let $T_i = \{v_{i,1}, \dots, v_{i,k}\}$. For $l = 1$, take a basis of a free isotropic direct summand of R^{2n}, for example $\{w_1, \dots, w_k\}$, such that $h(w_j, v_{1,m}) = \delta_{j,m}$, where $\delta_{j,m}$ is the Kronecker delta and choose T an extension of this basis to a basis of a maximal isotropic free subspace. Assume that the claim is true for $1 \le i \le l-1$. This means that there is a basis $\{u_1, \dots, u_n\}$ of a free isotropic summand of R^{2n}, in general position with T_i, $1 \le i \le l-1$. Let $\{x_1, \dots, x_k\}$ be a basis of a free isotropic summand of R^{2n} such that $h(x_j, v_{l,m}) = \delta_{j,m}$ and take $A = \prod E_{r,s}(a) \in E_n$ such that $Au_j = x_j$, $1 \le j \le k$ [7, 6.5, 7.1]. If $x_j := Au_j$ for $k+1 \le j \le n$, then $\{x_1, \dots, x_n\}$ is in general position with T_l. Set $B_i = (h(u_j, v_{i,m}))$, $1 \le i \le l-1$ and $B_l = (h(x_j, v_{l,m}))$. Let $B_i^{(k)}$ be the matrix obtained from B_i by deleting $(j_{1,i}, \dots, j_{n-k,i})$-th rows such that $f_i^{(k)} := \det(B_i^{(k)}) \in R^*$ for all i. Set $A(t) = \prod E_{r,s}(ta)$, $B_i(t) = (h(A(t)u_j, v_{i,m}))$, for $1 \le i \le l$ and let $B_i^{(k)}(t)$ be the matrix obtained by deleting $(j_{1,i}, \dots, j_{n-k,i})$-th rows of $B_i(t)$ and set $f_i^{(k)}(t) := \det(B_i^{(k)}(t)) \in R[t]$. Clearly $f_i^{(k)}(0) = f_i^{(k)}$ for $1 \le i \le l-1$ and $f_l^{(k)}(1) = f_l^{(k)}$. It is not difficult to see that there is a $t_1 \in R$ such that $f_i^{(k)}(t_1) \in R^*$ for $1 \le i \le l$ [12, 1.4, 1.5]. Take $W = \{A(t_1)u_1, \dots, A(t_1)u_n\}$.

The second part of the proposition follows from the exact sequence

$$0 \to W \cap V_1^{\perp} \to W \xrightarrow{\psi} R^k \to 0$$

with $\psi(w) := (h(w, v_{1,1}), \dots, h(w, v_{1,k}))$ and the fact that projective modules over local rings are free. $\square$

Let S be a non-empty set and $X \subseteq \mathcal{O}(S)$ [7, Sec. 4]. Let $C_k(X)$, $k \geq 0$, be the free $\mathbb{Z}$-module with the basis consisting of the k-simplices ($(k+1)$-frames) of X, $C_{-1}(X) = \mathbb{Z}$ and $C_k(X) = 0$ for $k \leq -2$. The family $C_*(X) := \{C_k(X)\}$ yields a chain complex with the differentials $\partial_0 : C_0(X) \to C_{-1}(X) = \mathbb{Z}$, $\sum_i n_i v_i \mapsto \sum_i n_i$ and $\partial_k = \sum_{i=0}^{k} (-1)^i d_i : C_k(X) \to C_{k-1}(X)$, $k \geq 1$, where $d_i((v_0, \dots, v_k)) = (v_0, \dots, \widehat{v_i}, \dots, v_k)$. The poset X is called n-acyclic if $\tilde{H}_k(X, \mathbb{Z}) := H_k(C_*(X)) = 0$ for all $0 \leq k \leq n$.

Lemma 2.3. *Let n, m be two natural numbers and $n \leq m$. If $n \geq k+1$ then for every $(v_1, \dots, v_k) \in \mathcal{U}(R^m)$ there is a $v \in R^n = \langle e_1, \dots, e_n \rangle$ such that $(v, v_1, \dots, v_k) \in \mathcal{U}(R^m)$.*

Proof. The proof is similar to the proof of lemma [7, 5.4], using the fact that $\mathrm{sr}(R) = 1$. $\square$

Theorem 2.4. *Let n, m be two natural numbers and $n \leq m$. Then the poset $\mathcal{O}(R^n) \cap \mathcal{U}(R^m)$ is $(n-2)$-acyclic and $\mathcal{O}(R^n) \cap \mathcal{U}(R^m)_w$ is $(n - |w| - 2)$-acyclic for every $w = (w_1, \dots, w_r) \in \mathcal{U}(R^m)$.*

Proof. Let $X = \mathcal{O}(R^n) \cap \mathcal{U}(R^m) = \mathcal{U}(R^n)$ and $\sigma = \sum_{i=1}^{l} n_i (v_0^i, \dots, v_k^i)$ be a cycle in $C_k(X)$, $k \leq n-2$. It is not difficult to see that there is a unimodular vector $v \in R^n$ such that $\{v, v_0^i, \dots, v_k^i\}$ is linearly independent, $1 \leq i \leq l$. If $\beta := \sum_{i=1}^{l} n_i (v, v_0^i, \dots, v_k^i) \in C_{k+1}(X)$, then $\partial_{k+1}(\beta) = \sigma$, so X is $(n-2)$-acyclic.

Let $Y = \mathcal{O}(R^n) \cap \mathcal{U}(R^m)_w$ and assume that $n - |w| - 2 \geq -1$. Let σ be a k-cycle in $C_k(Y)$ with $k \leq n - |w| - 2$. To prove the second part of the theorem it is sufficient to find a unimodular vector $v \in R^n$ such that $\{v, v_0^i, \dots, v_k^i, w_1, \dots, w_r\}$ is linearly independent, $1 \leq i \leq l$. The proof is by induction on l. The case $l = 1$ follows from 2.3. By induction assume that there are $u_1, u_2 \in R^n$ such that $(u_1, v_0^i, \dots, v_k^i, w_1, \dots, w_r) \in \mathcal{U}(R^m)$ for $1 \leq i \leq l-1$ and $(u_2, v_0^l, \dots, v_k^l, w_1, \dots, w_r) \in \mathcal{U}(R^m)$. Let $A = \prod E_{r,s}(a)$ be an element of the elementary group $E_n(R) \subseteq GL_n(R)$ such that $Au_1 = u_2$ and set $A(t) = \prod E_{r,s}(ta)$. Let B_i be the matrix whose columns are the vectors $u_1, v_0^i, \dots, v_k^i, w_1, \dots, w_r$ for $1 \leq i \leq l-1$, B_l the matrix whose columns are $u_2, v_0^l, \dots, v_k^l, w_1, \dots, w_r$ and $B_i(t)$ is the matrix whose columns are $A(t)u_1, v_0^i, \dots, v_k^i, w_1, \dots, w_r$, $1 \leq i \leq l$. The rest of the proof is similar to the proof of proposition 2.2. $\square$

Theorem 2.5. *The poset $\mathcal{IU}(R^{2n})$ is $(n-2)$-acyclic.*

Proof. If $n = 1$, then everything is trivial, so we assume that $n \geq 2$. Let $\sigma = \sum_{i=1}^{r} n_i v_i$ be a k-cycle. Thus v_i, $1 \leq i \leq r$, are isotropic $(k+1)$-frames with $k \leq n-2$. By 2.2, there is an isotropic n-frame w in general

position with v_i, $1 \le i \le r$. Set $W = \langle w \rangle$ and let E_σ be the set of all $(u_1, \dots, u_m, t_1, \dots, t_l) \in \mathcal{IU}(R^{2n})$ such that $m, l \ge 0$, $(u_1, \dots, u_m) \in \mathcal{U}(W)$, if $m \ge 1$, and for every $l \ge 1$ there exist an i such that $(t_1, \dots, t_l) \le v_i$. The poset E_σ satisfies the chain condition and $v_i \in E_\sigma$. It is sufficient to prove that E_σ is $(n-2)$-acyclic, because then $\sigma \in \partial_{k+1}(E_\sigma) \subseteq \partial_{k+1}(C_{k+1}(X))$. Let $F := E_\sigma$. Since $\mathcal{O}(W) \cap F = \mathcal{U}(W)$, by 2.4 the poset $\mathcal{O}(W) \cap F$ is $(n-2)$-acyclic. If $u \in F \backslash \mathcal{O}(W)$, then u is of the form $(u_1, \dots, u_m, t_1, \dots, t_l)$, $l \ge 1$. By 2.2, $\dim(V) = n - l$, where $V = W \cap \langle t_1 \dots t_l \rangle^\perp$. With all this we have

$$\mathcal{O}(W) \cap F_u = \mathcal{O}(V) \cap \mathcal{IU}(R^{2n})_{(u_1,\dots,u_m)} = \mathcal{O}(V) \cap \mathcal{U}(W)_{(u_1,\dots,u_m)}.$$

Again by 2.4, $\mathcal{O}(V) \cap \mathcal{U}(W)_{(u_1,\dots,u_m)}$ is $((n-l)-m-2)$-acyclic, so $\mathcal{O}(W) \cap F_u$ is $(n - |u| - 2)$-acyclic. Therefore F is $(n-2)$-acyclic [13, 2.13 (i)]. $\square$

Remark 1. (i) The concept of *being in general position* and the idea of the proof of 2.5 is taken from [9]. Because the details of the proof in [9] never appeared we wrote it down.

(ii) In fact Theorem 2.5 is true for every field $R \neq \mathbb{Z}/2\mathbb{Z}$. Let

$$\mathcal{IV}(R^{2n}) = \{V \subseteq R^{2n} : V \neq 0 \text{ and isotropic subspace}\}.$$

Define the map of the posets $f : \mathcal{IU}(R^{2n}) \to \mathcal{IV}(R^{2n})$, $v \mapsto \langle v \rangle$. As Vogtmann proved, [15, Thm. 1.6], $\mathcal{IV}(R^{2n})$ is $(n-2)$-connected (Vogtmann proved this for $G_n = O_{2n}(R)$, but her proof works without modification in our more general setting [2, p. 115]). On the one hand it is easy to see that $Link^+_{\mathcal{IV}(R^{2n})}(V) \simeq \mathcal{IV}(R^{2(n-\dim(V))})$, so it is $(n - \dim(V) - 2)$-connected and on the other hand $f/V = \mathcal{U}(V)$ which is $(\dim(V) - 2)$-connected [13, 2.6], hence defining the height function $\mathrm{ht}_{\mathcal{IV}(R^{2n})}(V) = \dim(V) - 1$ [7, section 2], one sees that $\mathcal{IU}(R^{2n})$ is $(n-2)$-connected [7, Thm. 3.8].

(iii) We expect that over a ring with no finite ring as a homomorphic image and finite unitary stable rank the poset $\mathcal{IU}(R^{2n})$ is $(n - \mathrm{usr}(R) - 1)$-connected. For this it is sufficient to prove 2.2 over such ring. For example Theorem 2.5, without any change in its proof, is true over a semi-local ring with infinite residue fields. Therefore the results of this note are also valid for these rings.

(iii) Using 2.5, (iii) and the same argument as in (ii) one can prove that over a semi-local ring with infinite residue fields, $\mathcal{IV}(R^{2n})$ is $(n-2)$-acyclic. Over an infinite field this gives much easier proof of Vogtmann's theorem mentioned in (ii).

(iv) Using a theorem of Van der Kallen [13, Thm. 2.6] and a similar arguments as (iii) we can generalize the Tits-Solomon theorem over a ring with stable range one (for example any Artinian ring). Let R be a ring with stable range one and consider the following poset, which we call it the Tits poset,

$$\mathcal{T}(R^n) = \{V \subseteq R^n : V \text{ free summand of } R^n, V \neq 0, R^n\}.$$

Let $X = \mathcal{U}(R^n)_{\le n-1}$ and consider the poset map $g : X \to \mathcal{T}(R^n)$, $v \mapsto \langle v \rangle$. By induction and a similar argument as in (ii), using the fact that X is

$(n-3)$connected, one can prove that $\mathcal{T}(R^n)$ is $(n-3)$-connected (note that any stably free projective module of rank ≥ 1 is free). We leave the details of the proof to the interested readers.

Definition 2.6. Define $\underline{\mathcal{U}}(R^n) = \{(\langle v_1\rangle, \dots, \langle v_k\rangle) : (v_1, \dots, v_k) \in \mathcal{U}(R^n)\}$ and $\underline{\mathcal{IU}}(R^{2n}) = \{(\langle v_1\rangle, \dots, \langle v_k\rangle) : (v_1, \dots, v_k) \in \mathcal{IU}(R^{2n})\}$.

Theorem 2.7. *Let n, m be two natural numbers and $n \leq m$. Then the poset $\mathcal{O}(\mathbb{P}^{n-1}) \cap \underline{\mathcal{U}}(R^m)$ is $(n-2)$-acyclic, the poset $\mathcal{O}(\mathbb{P}^{n-1}) \cap \underline{\mathcal{U}}(R^m)_w$ is $(n-|w|-2)$-acyclic for every $w \in \underline{\mathcal{U}}(R^m)$ and the poset $\underline{\mathcal{IU}}(R^{2n})$ is $(n-2)$-acyclic.*

Proof. The proof is similar to the proof of 2.4 and 2.5. □

3. The spectral sequence

In this section, k will be a field, S_i a k-algebra, $i \in \mathbb{N}$, $S_i^{\otimes n} := S_i \otimes_k \cdots \otimes_k S_i$ (n-times) and $V_n(S_i) := (S_i^{\otimes n})^{\Sigma_n}$, where Σ_n is the symmetric group of degree n.

Lemma 3.1. *Let $\varphi_i : R \to S_i$ be a ring homomorphism, $i = 1, \dots, d$. Consider the action of R^* on $\bigotimes_{i=1}^d S_i^{\otimes n_i}$ and $\bigotimes_{i=1}^d V_{n_i}(S_i)$ as*

$$r \bigotimes_{i=1}^{d} (a_{1,i} \otimes \cdots \otimes a_{n_i,i}) = \bigotimes_{i=1}^{d} (\varphi_i(r)^{t_i} a_{1,i} \otimes \cdots \otimes \varphi_i(r)^{t_i} a_{n_i,i}),$$

where $t_i \geq 1$. Then $H_0(R^*, \bigotimes_{i=1}^d S_i^{\otimes n_i}) = H_0(R^*, \bigotimes_{i=1}^d V_{n_i}(S_i)) = 0$.

Proof. The proof is similar to the proof of [8, 1.5] and [8, 1.6] with minor generalization. If $B := \bigotimes_{i=1}^d S_i^{\otimes n_i}$, then $H_0(R^*, B) = B/I$, where I is the ideal of B generated by the elements $\bigotimes_{i=1}^d (\varphi_i(r)^{t_i} \otimes \cdots \otimes \varphi_i(r)^{t_i}) - 1$. Consider the collection $\{\psi_1^{(j_i)}, \dots, \psi_{t_i}^{(j_i)}\}$, $i = 1, \dots, d$, $1 \leq j_1 \leq n_1$ and $\sum_{i=1}^{m-1} n_i < j_m \leq \sum_{i=1}^m n_i$ for $m \geq 2$, of homomorphisms $R \to B/I$ given by $\psi_l^{(j_i)}(r) = 1 \otimes \cdots \otimes \varphi_i(r) \otimes \cdots \otimes 1 \mod I$, with $\varphi_i(r)$ in the j_i-th position. For simplicity we denote this collection by $\{\psi_l : 1 \leq l \leq \sum_{i=1}^d t_i n_i\}$. If I is a proper ideal, we obtain a collection of ring homomorphisms ψ_l such that $\prod_l \psi_l(r) = 1$ for every $r \in R^*$, but this is impossible [8, Cor. 1.3, Lem. 1.4]. Thus $I = B$ and therefore $H_0(R^*, \bigotimes_{i=1}^d S_i^{\otimes n_i}) = 0$. For the proof of the second part let $l_1^{(i)}, \dots, l_{s_i}^{(i)}$, $i = 1, \dots, d$, be the natural numbers such that $\sum_{j=1}^{s_i} l_j^{(i)} = n_i$, and denote by $V_{n_i}^{l_1^{(i)}, \dots, l_{s_i}^{(i)}}$ the subspace of $V_{n_i}(S_i)$ generated by the elements of the form

$$y_{c,l^{(i)},i} := \sum_{\delta \in \Sigma_{n_i}/\Sigma_{l_1^{(i)}} \times \cdots \times \Sigma_{l_{s_i}^{(i)}}} (\underbrace{c_1^{(i)} \otimes \cdots \otimes c_1^{(i)}}_{l_1^{(i)}} \otimes \cdots \otimes \underbrace{c_{s_i}^{(i)} \otimes \cdots \otimes c_{s_i}^{(i)}}_{l_{s_i}^{(i)}})^{\delta}.$$

Clearly $V_{n_i}^{l_1^{(i)},\dots,l_{s_i}^{(i)}}$ is an R^*-invariant subspace of $V_{n_i}(S_i)$ and $V_{n_i}(S_i) = \sum_{l_1^{(i)}+\dots+l_{s_i}^{(i)}=n_i} V_{n_i}^{l_1^{(i)},\dots,l_{s_i}^{(i)}}$. Let $V_{n_i}^{(j)}(S_i) = \sum_{s_i\ge n_i-j} V_{n_i}^{l_1^{(i)},\dots,l_{s_i}^{(i)}}$ and set

$$T_h := \sum_{h_1+\dots+h_d=h} V_{n_1}^{(h_1)}(S_1)\otimes\dots\otimes V_{n_d}^{(h_d)}(S_d).$$

It is not difficult to see that if $\sum_{i=1}^d n_i - s_i = h$ and $l_1^{(i)}+\dots+l_{s_i}^{(i)} = n_i$, then

$$\bigotimes_{i=1}^d S_i^{\otimes s_i} \to T_h/T_{h-1},$$

$$\bigotimes_{i=1}^d c_1^{(i)}\otimes\dots\otimes c_{s_i}^{(i)} \mapsto y_{c,l^{(1)},1}\otimes\dots\otimes y_{c,l^{(d)},d} \mod T_{h-1}$$

is multilinear, so it gives an R^*-equivariant homomorphism. In this way we obtain an R^*-equivariant epimorphism

$$\coprod_{n_1-s_1+\dots+n_d-s_d=h} \bigotimes_{i=1}^d S_i^{\otimes s_i} \quad\to\quad T_h/T_{h-1}.$$

Since the functor H_0 is right exact, by applying the first part of the lemma we get $H_0(R^*, T_h/T_{h-1}) = 0$. By induction on h we prove that $H_0(R^*, T_h) = 0$. If $h = 0$, then $T_0 = \bigotimes_{i=1}^d V_{n_i}^{(0)}(S_i)$ and $\bigotimes_{i=1}^d S_i^{\otimes n_i} \to T_0$ is surjective, so $H_0(R^*, T_0) = 0$. By induction and applying the functor H_0 to the short exact sequence $0\to T_{h-1}\to T_h\to T_h/T_{h-1}\to 0$, we see that $H_0(R^*, T_h) = 0$. $\square$

Lemma 3.2. *Let P_i and Q_i be S_i-modules for $i = 1,\dots,d$. Then $\bigotimes_{i=1}^d \bigwedge^{n_{i,1}} P_i\otimes_k V_{n_{i,2}}(Q_i)$ has a natural structure of $\bigotimes_{i=1}^d V_{n_i}(S_i)$-module, where $n_i = n_{i,1}+n_{i,2}$. Moreover for all $l\ge 0$*

$$H_l(R^*, \bigotimes_{i=1}^d \bigwedge^{n_{i,1}} P_i\otimes_k V_{n_{i,2}}(Q_i)) = 0.$$

Proof. The first part follows immediately from [8, Lem. 1.7] and the second part follows from 3.1 and [8, Lem. 1.8]. $\square$

Let B be a k-vector space and let $\Gamma(B)$ be the algebra of divided powers of B, which is a graded commutative algebra concentrated in even degrees and endowed with a system of divided powers with $\Gamma_{2n}(B) = V_n(B)$ (see ([1, Chap. V, No. 6] and [8, §1] for more details). The homology of an abelian group A with rational coefficients coincides with exterior powers: $H_p(A,\mathbb{Q}) = \bigwedge^p(A\otimes\mathbb{Q})$. The homology with coefficients in the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is more complicated. The ring $H_*(A,\mathbb{F}_p)$ has a canonical structure of divided powers [1, Chap. V, Example 6.5.4]. Moreover, $H_1(A,\mathbb{F}_p) = A/pA$ and there is an exact sequence

$$0\to \bigwedge^2(A/pA)\to H_2(A,\mathbb{F}_p)\xrightarrow{\beta} {}_pA\to 0.$$

Any choice of a section for β gives a homomorphism $\varphi : {}_pA \to H_2(A, \mathbb{F}_p)$, which by the property of the algebra Γ, uniquely extends to an $\mathbb{F}_p$-algebra homomorphism $\bigwedge^2(A/pA) \otimes_k \Gamma({}_pA) \to H_*(A, \mathbb{F}_p)$, thus giving rise to an isomorphism of graded $\mathbb{F}_p$-algebras [1, Chap. V, Thm. 6.6], [10, §8, Prop. 3]. We identify $H_j(A, \mathbb{F}_p)$ with $\coprod_i \bigwedge^{j-2i}(A/pA) \otimes_k \Gamma_{2i}({}_pA)$ and introduce a filtration on $H_j(A, \mathbb{F}_p)$ setting

$$H_j^{(r)} = \coprod_{i \le r} \bigwedge^{j-2i}(A/pA) \otimes_k \Gamma_{2i}({}_pA).$$

This filtration does not depend on our choice of section φ and successive factors $H_j^{(r)}/H_j^{(r-1)}$ are canonically isomorphic to $\bigwedge^{j-2r}(A/pA) \otimes_k \Gamma_{2r}({}_pA)$.

Theorem 3.3. *Let M_i be a T_i-module and let $R \to T_i$ be a ring homomorphism. Consider the action of R^* on M_i given by $r \cdot m = \varphi_i(r)^{t_i} m$, where $t_i \ge 1$. If k is a prime field, then $H_l(R^*, \bigotimes_{i=1}^d H_{l_i}(M_i)) = 0$ for $l \ge 0$, where $l_i > 0$ for some i.*

Proof. Let $P_i = M_i \otimes_{\mathbb{Z}} k$ and $S_i = T_i \otimes_{\mathbb{Z}} k$. If $k = \mathbb{Q}$, then $H_{l_i}(M_i) = \bigwedge^{l_i} P_i$ and if $k = \mathbb{F}_p$ for some prime number p, then we can find an R^*-invariant filtration of $\bigotimes_{i=1}^d H_{l_i}(M_i)$ whose successive factors are isomorphic to $\bigotimes_{i=1}^d \bigwedge^{j_i - 2m_i} P_i \otimes_k \Gamma_{2m_i}(Q_i)$ for some j_i and m_i, where $Q_i = {}_p(P_i \otimes_{\mathbb{Z}} k)$. Note that P_i and Q_i are S_i-modules. Then both cases follow from 3.2. □

Let $\overline{\sigma_2} = (\langle e_1 \rangle, \langle e_3 \rangle) \in \underline{\mathcal{IU}}(R^{2n})$. The elements of $\mathrm{Stab}_{G_n}(\overline{\sigma_2}) = \{B \in G_n : B\overline{\sigma_2} = \overline{\sigma_2}\}$ are of the form

$$\begin{pmatrix} a_1 & * & 0 & * & * & * \\ 0 & \overline{a_1}^{-1} & 0 & 0 & 0 & 0 \\ 0 & * & a_2 & * & * & * \\ 0 & 0 & 0 & \overline{a_2}^{-1} & 0 & 0 \\ 0 & * & 0 & * & & \\ 0 & * & 0 & * & & A \end{pmatrix},$$

where $a_i \in R^*$ and $A \in G_{n-2}$. Let $N_{n,2}$ and $L_{n,2}$ be the subgroups of $\mathrm{Stab}_{G_n}(\overline{\sigma_2})$ of elements of the form

$$\begin{pmatrix} 1 & * & 0 & * & * & * \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & * & 0 & * & & \\ 0 & * & 0 & * & & I_{2(n-2)} \end{pmatrix}, \quad \begin{pmatrix} a_1 & * & 0 & * & * & * \\ 0 & \overline{a_1}^{-1} & 0 & 0 & 0 & 0 \\ 0 & * & a_2 & * & * & * \\ 0 & 0 & 0 & \overline{a_2}^{-1} & 0 & 0 \\ 0 & * & 0 & * & & \\ 0 & * & 0 & * & & I_{2(n-2)} \end{pmatrix}$$

respectively. It is a matter of an easy calculation to see that the elements of the group $N'_{n,2} = [N_{n,2}, N_{n,2}]$ are of the form

$$\begin{pmatrix} 1 & r & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\epsilon^{-1}\overline{t} & 1 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & I_{2(n-2)} \end{pmatrix},$$

where $r,\ s \in \Lambda = \{r \in R : \epsilon^{-1}\overline{r} = -r\}$ and $t \in R$. In general one can define $N_{n,p}$, $L_{n,p}$ and $N'_{n,p}$ for all p, $1 \le p \le n$, in a similar way. Embed $R^{*p} \times G_{n-p}$ in $\mathrm{Stab}_{G_n}(\overline{\sigma_p})$ as $\mathrm{diag}(a_1, \dots, a_p, A) \mapsto \mathrm{diag}(\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1}^{-1} \end{pmatrix}, \dots, \begin{pmatrix} a_p & 0 \\ 0 & \overline{a_p}^{-1} \end{pmatrix}, A)$.

Theorem 3.4. *Let $\overline{\sigma_p} = (\langle e_1\rangle, \langle e_3\rangle, \dots, \langle e_{2p-1}\rangle) \in \underline{\mathcal{IU}}(R^{2n})$. Then the inclusion $R^{*p} \times G_{n-p} \to \mathrm{Stab}_{G_n}(\overline{\sigma_p})$ induces the isomorphism between the homology groups $H_i(R^{*p} \times G_{n-p}) \to H_i(\mathrm{Stab}_{G_n}(\overline{\sigma_p}))$ for all i.*

Proof. It is sufficient to prove the theorem when k is a prime field. Fix a natural number p, $1 \le p \le n$, and set $N = N_{n,p}$, $L = L_{n,p}$, $N' = N'_{n,p}$ and $T = \mathrm{Stab}_{G_n}(\overline{\sigma_p})$. The extensions $1 \to N' \to L \to L/N' \to 1$ and $1 \to N/N' \to L/N' \to L/N \to 1$ give the Lyndon-Hochschild-Serre spectral sequences

$$E^2_{p,q} = H_p(L/N', H_q(N')) \Rightarrow H_{p+q}(L),$$
$$E^2_{p',q'} = H_{p'}(L/N, H_{q'}(N/N', H_q(N'))) \Rightarrow H_{p'+q'}(L/N', H_q(N')),$$

respectively. Since $L/N \simeq R^{*p}$ and N/N' acts trivially on N', $E^2_{p',q'} = H_{p'}(R^{*p}, H_{q'}(N/N') \otimes_k H_q(N'))$. It is not difficult to see that $N/N' \simeq R^h$ and $N' \simeq R^l \times \Lambda^m$ for some h, l, m and the action of R_1^* on N/N' and N' is linear-diagonal and quadratic-diagonal respectively. Again the extension $1 \to R_1^* \to R^{*p} \to R^{*p}/R_1^* \to 1$ (R_1^* embeds in R^{*p} diagonally) gives

$$E^2_{r,s} = H_r(R^{*p}/R_1^*, H_s(R_1^*, M)) \Rightarrow H_{r+s}(R^{*p}, M),$$

where $M = H_{q'}(N/N') \otimes_k H_q(N')$. Since the homology functor commutes with the direct sum functor,

$$H_s(R_1^*, M) \simeq \bigoplus_{i=0}^{q} H_s(R_1^*, H_{q'}(R^h) \otimes_k H_i(R^l) \otimes_k H_{q-i}(\Lambda^m)),$$

where the action of R_1^* on R^h, R^l and Λ^m is linear-diagonal, quadratic-diagonal and quadratic-diagonal respectively. By theorem 3.3, $H_s(R_1^*, M) = 0$ for $s \ge 0$ and $q > 0$ or $q' > 0$. This shows that $E^2_{p',q'} = 0$ for $p' \ge 0$ and $q > 0$ or $q' > 0$. Therefore $H_{p'}(L/N', H_q(N')) = 0$ for $p' \ge 0$ and $q > 0$.

Hence $E^2_{p,q} = 0$ for $p \geq 1$ and $q > 0$. By the convergence of the spectral sequence we get

$$H_p(L) \xrightarrow{\simeq} H_p(L/N'). \tag{1}$$

The extension $1 \to N/N' \to L/N' \to L/N \to 1$ gives

$$E^2_{i,j} = H_i(L/N, H_j(N/N')) \Rightarrow H_{i+j}(L/N').$$

and by a similar approach to (1),

$$H_i(R^{*p}) \xrightarrow{\simeq} H_i(L/N'). \tag{2}$$

From the embedding $R^{*p} \to L$, (1) and (2) we get the isomorphism $H_i(R^{*p}) \xrightarrow{\simeq} H_i(L)$, $i \geq 0$. The commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & R^{*p} & \to & R^{*p} \times G_{n-p} & \to & G_{n-p} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & L & \to & T & \to & G_{n-p} & \to & 1
\end{array}$$

gives the map of the spectral sequences

$$\begin{array}{ccc}
E^2_{p,q} = H_p(G_{n-p}, H_q(R^{*p})) & \Rightarrow & H_{p+q}(R^{*p} \times G_{n-p}) \\
\Big\downarrow & & \Big\downarrow \\
E'^2_{p,q} = H_p(G_{n-p}, H_q(L)) & \Rightarrow & H_{p+q}(T).
\end{array}$$

By what we proved in the above we have the isomorphism $E^2_{p,q} \simeq E'^2_{p,q}$. This gives an isomorphism on the abutments and so $H_i(R^{*p} \times G_{n-p}) \simeq H_i(T)$. $\square$

Theorem 3.5. *There is a first quadrant spectral sequence converging to zero with*

$$E^1_{p,q}(n) = \begin{cases} H_q(R^{*p} \times G_{n-p}) & \text{if } 0 \leq p \leq n \\ H_q(G_n, H_{n-1}(X_n)) & \text{if } p = n+1 \\ 0 & \text{if } p \geq n+2 \end{cases},$$

where $X_n = \underline{\mathcal{IU}}(R^{2n})$.

For $1 \leq p \leq n$ *the differential* $d^1_{p,q}(n)$ *equals* $\sum_{i=1}^{p} (-1)^{i+1} H_q(\alpha_{i,p})$, *where* $\alpha_{i,p} : R^{*p} \times G_{n-p} \to R^{*p-1} \times G_{n-p+1}$, $\mathrm{diag}(a_1, \dots, a_p, A) \mapsto \mathrm{diag}(a_1, \dots, \widehat{a_i}, \dots, a_p, \begin{pmatrix} a_i & 0 & 0 \\ 0 & \overline{a_i}^{-1} & 0 \\ 0 & 0 & A \end{pmatrix})$.

In particular for $0 \leq p \leq n$, $d^1_{p,0}(n) = \begin{cases} \mathrm{id}_k & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even} \end{cases}$, *so* $E^2_{p,0} = 0$ *for* $0 \leq p \leq n-1$.

Proof. Let $C'_l(X_n)$ be the k-vector space with the basis consisting of l-simplices (isotropic $(l+1)$-frames) of X_n. Since X_n is $(n-2)$-acyclic, Theorem 2.7, we get an exact sequence

$$0 \leftarrow k \leftarrow C'_0(X_n) \leftarrow C'_1(X_n) \leftarrow \cdots \leftarrow C'_{n-1}(X_n) \leftarrow H_{n-1}(X_n, k) \leftarrow 0.$$

Call this exact sequence L_*: $L_0 = k, L_i = C'_{i-1}(X_n)$, $1 \leq i \leq n$, $L_{n+1} = H_{n-1}(X_n, k)$ and $L_i = 0$ for $i \geq n+2$. Let $F_* \to k$ be a resolution of k by free

(left) G_n-modules and consider the bicomplex $C_{*,*} = L_* \otimes_{G_n} F_*$. Here we convert the left action of G_n on L_* into a right action with $vg := g^{-1}v$. By the general theory of the spectral sequence for a bicomplex we have $E^1_{p,q}(I) = H_q(C_{p,*}) = H_q(L_p \otimes_{G_n} F_*)$ and $E^1_{p,q}(II) = H_q(C_{*,p}) = H_q(L_* \otimes_{G_n} F_p)$. Since F_p is a free G_n-module, $L_* \otimes_{G_n} F_p$ is exact and this shows that $E^1_{p,q}(II) = 0$. Therefore $E^1_{p,q}(n) := E^1_{p,q}(I)$ converges to zero. If $p = 0$, then $E^1_{0,q}(n) = H_q(k \otimes_{G_n} F_*) = H_q(G_n)$. The group G_n acts transitively on the l-frames of X_n, $1 \le l \le n$, so by the Shapiro lemma [1, Chap. III, 6.2], $L_p \otimes_{G_n} F_* \simeq k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_p})} F_*$ and thus $E^1_{p,q}(n) = H_q(\mathrm{Stab}_{G_n}(\overline{\sigma_p}))$, $1 \le p \le n$. By 3.4, $E^1_{p,q}(n) = H_q(R^{*p} \times G_{n-p})$ for $0 \le p \le n$, hence $E^1_{p,q}(n)$ is of the form that we mentioned. Now we look at the differential $d^1_{p,q}(n) : E^1_{p,q}(n) \to E^1_{p-1,q}(n)$, $1 \le p \le n$; $d^1_{1,q}(n)$ is induced by $C'_0(X_n) \otimes_{G_n} F_* \to k \otimes_{G_n} F_*$, $\overline{\sigma_1} \otimes x \mapsto 1 \otimes x$. Considering the isomorphism $k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_1})} F_* \to C'_0(X) \otimes_{G_n} F_*$, $1 \otimes x \mapsto \overline{\sigma_1} \otimes x$, one sees that $d^1_{1,q}(n)$ is induced by $k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_1})} F_* \to k \otimes_{G_n} F_*$. This shows that $d^1_{1,q}(n)$ is the map $H_q(\mathrm{Stab}_{G_n}(\overline{\sigma_1})) \to H_q(G_n)$ induced by the inclusion map, therefore $d^1_{1,q}(n) = H_q(\mathrm{inc}) : H_q(\mathrm{inc}) : H_q(R^* \times G_{n-1}) \to H_q(G_n)$. For $2 \le p \le n$, $d^1_{p,q}(n)$ is induced by the map $\sum_{i=1}^{p} (-1)^{i+1} d_i : L_p \to L_{p-1}$, where d_i deletes the i-th component of the isotropic p-frames. Let $g_{i,p}$ be the permutation matrix such that $(e_{2h-1}, e_{2h}) g_{i,p}^{-1} = (e_{2h-1}, e_{2h})$, $1 \le h \le i-1$, $(e_{2i-1}, e_{2i}) g_{i,p}^{-1} = (e_{2p-1}, e_{2p})$ and $(e_{2l-1}, e_{2l}) g_{i,p}^{-1} = (e_{2l-3}, e_{2l-2})$, $i+1 \le l \le p$, where $vg^{-1} := gv$ for $v \in R^{2n}$. It is easy to see that $d_i(\overline{\sigma_p}) = \overline{\sigma_{p-1}} g_{i,p}$, and so $\partial(\overline{\sigma_p}) = \sum_{i=1}^{p} (-1)^{i+1} \overline{\sigma_{p-1}} g_{i,p}$. Consider $d_i \otimes \mathrm{id}_{F_*} : L_p \otimes_{G_n} F_* \to L_{p-1} \otimes_{G_n} F_*$, $\overline{\sigma_p} \otimes x \mapsto d_i(\overline{\sigma_p}) \otimes x$. Let $\mathrm{inn}_{g_{i,p}} : G_n \to G_n$, $g \mapsto g_{i,p} g g_{i,p}^{-1}$ and $l_{g_{i,p}} : F_* \to F_*$, $x \mapsto g_{i,p} x$. It is easy to see that $l_{g_{i,p}}$ is an $\mathrm{inn}_{g_{i,p}}$-homomorphism, and $d_i \otimes \mathrm{id}_{F_*}$ induces the map $k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_p})} F_* \to k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_{p-1}})} F_*$, $1 \otimes x \mapsto 1 \otimes l_{g_{i,p}}(x)$. This shows that d_i induces $H_q(\mathrm{inn}_{g_{i,p}}) : H_q(\mathrm{Stab}_{G_n}(\overline{\sigma_p})) \to H_q(\mathrm{Stab}_{G_n}(\overline{\sigma_{p-1}}))$ and hence the map $H_q(\mathrm{inn}_{g_{i,p}}) : H_q(R^{*p} \times G_{n-p}) \to H_q(R^{*p-1} \times G_{n-p+1})$. Set $\alpha_{i,p} = \mathrm{inn}_{g_{i,p}}$. Since G_n acts transitively on the generators of $C'_p(X_n)$, $E^1_{*,0}(n)$ is of the following form

$$0 \leftarrow k \leftarrow k \leftarrow k \leftarrow \cdots \leftarrow k \leftarrow H_0(G_n H_{n-1}(X_n, k)) \leftarrow 0,$$

where $d^1_{p,0}(n) = \begin{cases} \mathrm{id}_k & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even} \end{cases}$. Clearly $E^2_{p,0}(n) = 0$ if $0 \le p \le n-1$. $\square$

Remark 2. In fact $E^2_{n,0}(n) = 0$. For a proof see the proof of theorem 4.3.

4. Stability theorem

To prove the homology stability result we have to study the spectral sequence that we obtained in theorem 3.5.

Lemma 4.1. *Let $n \geq 1$, $l \geq 0$ be integer numbers such that $n-1 \geq l$. Let $H_q(\mathrm{inc}) : H_q(G_{n-2}) \to H_q(G_{n-1})$ be surjective if $0 \leq q \leq l-1$. Then the following conditions are equivalent;*
(i) *$H_l(\mathrm{inc}) : H_l(G_{n-1}) \to H_l(G_n)$ is surjective,*
(ii) *$H_l(\mathrm{inc}) : H_l(R^* \times G_{n-1}) \to H_l(G_n)$ is surjective.*

Proof. For $n = 1$ every thing is easy so let $n \geq 2$. By the Künneth theorem [5, Chap. V, §10, Thm. 10.1] we have $H_l(R^* \times G_{n-1}) = S_1 \oplus S_2$, where $S_1 = H_l(G_{n-1})$ and $S_2 = \bigoplus_{i=1}^{l} H_i(R^*) \otimes_k H_{l-i}(G_{n-1})$. The case (i)⇒(ii) is trivial. To prove (ii)⇒(i) it is sufficient to prove that $\tau_1(S_2) \subseteq \tau_1(S_1)$, where $\tau_1 = H_l(\mathrm{inc})$. From $i \geq 1$ and $n-1 \geq l$, we have $n-2 \geq l-1 \geq l-i$, so by hypothesis $H_{l-i}(\mathrm{inc}) : H_{l-i}(G_{n-2}) \to H_{l-i}(G_{n-1})$ is surjective, $1 \leq i \leq l$. Consider the following diagram

$$\begin{tikzcd}
H_i(R^*) \otimes_k H_{l-i}(G_{n-1}) \arrow[r, "\beta_1"] & H_l(R^* \times G_{n-1}) \arrow[r, "\tau_1"] & H_l(G_n) \\
H_i(R^*) \otimes_k H_{l-i}(G_{n-2}) \arrow[u, "\alpha_1"] \arrow[r, "\beta_2"] & H_l(R^* \times G_{n-2}) \arrow[r, "\tau_1'"] & H_l(G_{n-1}), \arrow[u, "\alpha_2"]
\end{tikzcd}$$

where β_j is the shuffle product, $j = 1, 2$ [1, Chap. V, Sec. 5], $\alpha_1 = \mathrm{id} \otimes H_{l-i}(\mathrm{inc})$ is surjective and $\alpha_2 = H_l(\mathrm{inc})$. By giving an explicit description of the above maps we prove that this diagram is commutative. For this purpose we use the the bar resolution of a group [1, Chap. I, Sec. 5]. If $x = \sum [a_1|\dots|a_i] \otimes [A_1|\dots|A_{l-i}] \in H_i(R^*) \otimes_k H_{l-i}(G_{n-2})$, then

$$x \overset{\alpha_1}{\longmapsto} \sum [a_1|\dots|a_i] \otimes [\mathrm{diag}(I_2, A_1)|\dots|\mathrm{diag}(I_2, A_{l-i})]$$
$$\overset{\tau_1 \circ \beta_1}{\longmapsto} \sum \sum_{\delta} \mathrm{sign}(\delta)[\dots|\mathrm{diag}(a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-1)})|\dots|\mathrm{diag}(I_4, A_{\delta(j')})|\dots]$$

and

$$\begin{aligned} x \overset{\tau_1' \circ \beta_2}{\longmapsto} & \sum \sum_{\delta} \mathrm{sign}(\delta)[\dots|\mathrm{diag}(a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-2)})|\dots| \\ & \qquad\qquad \mathrm{diag}(I_2, A_{\delta(j')})|\dots] \\ \overset{\alpha_2}{\longmapsto} & \sum \sum_{\delta} \mathrm{sign}(\delta)[\dots|\mathrm{diag}(I_2, a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-2)})|\dots| \\ & \qquad\qquad \mathrm{diag}(I_4, A_{\delta(j')})|\dots]. \end{aligned}$$

See [5, Chap. VIII, §8] for more details about the shuffle product. Let $P \in G_n$ be the permutation matrix that permutes the first and second columns with third and forth columns respectively and let $\mathrm{inn}_P : G_n \to G_n$, $A \mapsto PAP^{-1} = PAP$. It is well known that $H_q(\mathrm{inn}_P) = \mathrm{id}_{H_q(G_n)}$ [1, Chap. II, §8], hence

$$\begin{aligned} H_l(\mathrm{inn}_P)([\dots|\mathrm{diag}(I_2, a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-2)})|\dots|\mathrm{diag}(I_2, I_2, A_{\delta(j')})|\dots]) \\ = [\dots|\mathrm{diag}(a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_2, I_{2(n-2)})|\dots|\mathrm{diag}(I_2, I_2, A_{\delta(j')})|\dots]. \end{aligned}$$

This shows that the above diagram is commutative. Therefore $\tau_1(S_2) \subseteq \tau_1(S_1)$. $\square$

Lemma 4.2. *Let $n \geq 2$, $l \geq 0$ be integer numbers such that $n-1 \geq l+1$. Let $H_q(\mathrm{inc}) : H_q(G_{m-1}) \to H_q(G_m)$ be isomorphism for $m = n-1, n-2$ and $0 \leq q \leq \min\{l-1, m-2\}$. Then the following conditions are equivalent;*
(i) *$H_l(\mathrm{inc}) : H_l(G_{n-1}) \to H_l(G_n)$ is bijective,*
(ii) *$H_l(R^{*2} \times G_{n-2}) \xrightarrow{\tau_2} H_l(R^* \times G_{n-1}) \xrightarrow{\tau_1} H_l(G_n) \to 0$ is exact, where $\tau_1 = H_l(\mathrm{inc})$ and $\tau_2 = H_l(\alpha_{1,2}) - H_l(\alpha_{2,2})$.*

Proof. Let $H_l(R^* \times G_{n-1}) = S_1 \oplus S_2$, where S_1 and S_2 are as in the proof of lemma 4.1 and $H_l(R^{*2} \times G_{n-2}) = \bigoplus_{h=1}^4 T_h$, where

$$T_1 = H_l(G_{n-2}),$$
$$T_2 = \bigoplus_{i=1}^{l} H_i(R^* \times 1) \otimes_k H_{l-i}(G_{n-2}),$$
$$T_3 = \bigoplus_{i=1}^{l} H_i(1 \times R^*) \otimes_k H_{l-i}(G_{n-2}),$$
$$T_4 = \bigoplus_{i+j \leq l} H_i(R^* \times 1) \otimes_k H_j(1 \times R^*) \otimes_k H_{l-i-j}(G_{n-2}).$$

Set $\sigma_1^{(2)} = H_l(\alpha_{1,2})$ and $\sigma_2^{(2)} = H_l(\alpha_{2,2})$. First (i)$\Rightarrow$(ii). The surjectivity of τ_1 is trivial. Let $(x, v) \in S_1 \oplus S_2$ such that $\tau_1((x, v)) = 0$. The relations $n-1 \geq l+1$ and $i \geq 1$ imply that $n-3 \geq l-1 \geq l-i$ and hence $H_{l-i}(G_{n-2}) \to H_{l-i}(G_{n-1})$ is bijective, so there exists $w \in T_2$ such that $-\sigma_1^{(2)}(w) = v$. If $y = (0, w, 0, 0) \in \bigoplus_{h=1}^4 T_h$, then $\tau_2(y) = (\sigma_2^{(2)}(w), -\sigma_1^{(2)}(w)) = (\sigma_2^{(2)}(w), v)$. Since $\tau_1 \circ \tau_2 = 0$, $\tau_1(\sigma_2^{(2)}(w)) = -\tau_1(v)$. Combining this with $\tau_1(x) = -\tau_1(v)$, we obtain $\tau_1(\sigma_2^{(2)}(w)) = \tau_1(x)$. By injectivity of $H_l(\mathrm{inc})$, $\sigma_2^{(2)}(w) = x$, thus $\tau_2(y) = (x, v)$. This shows that the complex is exact. The proof of (ii)$\Rightarrow$(i) is more difficult. The surjectivity of $\tau_1 = H_l(\mathrm{inc})$ follows from lemma 4.1. Let $x \in \ker(H_l(G_{n-1}) \to H_l(G_n))$, then $(x, 0) \in \ker(\tau_1)$. By exactness of the complex there is a $y = (0, y_2, y_3, y_4) \in \bigoplus_{h=1}^4 T_h$ such that $\tau_2(y) = (x, 0)$ (one should notice that $\tau_2(T_1) = 0$). First we prove that we can assume $y_4 = 0$. If $n = 2$, then $l \leq 1$ and so $T_4 = 0$. Therefore we may assume $n \geq 3$. Consider the summand

$$U = \bigoplus_{i,j \geq 1} H_i(R^* \times 1 \times 1) \otimes_k H_j(1 \times R^* \times 1) \otimes_k H_{l-i-j}(G_{n-3})$$

of $H_l(R^{*3} \times G_{n-3})$ and set $\tau_3 := d_{3,l}^1(n) = \sigma_1^{(3)} - \sigma_2^{(3)} + \sigma_3^{(3)}$, where $\sigma_i^{(3)} = H_l(\alpha_{i,3})$. It is easy to see that $\sigma_3^{(3)}(U) \subseteq T_4$ and $-\sigma_2^{(3)} + \sigma_1^{(3)}(U) \subseteq T_2 \oplus T_3$. By assumption, $\sigma_3^{(3)}|_U : U \to T_4$ is an isomorphism. If $\sigma_3^{(3)}(z) = y_4$, then $y - \tau_3(z) = (0, y_2', y_3', 0)$ and $\tau_2(y - \tau_3(z)) = (x, 0)$. So we can assume that

$y = (0, y_2, y_3, 0)$. Let

$$y_2 = (\sum [a_1|\dots|a_i] \otimes [A_1|\dots|A_{l-i}])_{1\le i\le l}$$
$$y_3 = (\sum [b_1|\dots|b_i] \otimes [B_1|\dots|B_{l-i}])_{1\le i\le l}.$$

By an explicit computation

$$\tau_2(y) = (\sigma_1^{(2)}(y_2) - \sigma_2^{(2)}(y_3), -\sigma_2^{(2)}(y_2) + \sigma_1^{(2)}(y_3)).$$

This shows that $x = \sigma_1^{(2)}(y_2) - \sigma_2^{(2)}(y_3)$ is equal to

$$\sum_{i=1}^{l}\sum\sum_{\delta} \mathrm{sign}(\delta)[\dots|\mathrm{diag}(a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-1)})|\dots|\mathrm{diag}(I_2, A_{\delta(j')})|\dots]$$
$$-\sum_{i=1}^{l}\sum\sum_{\delta} \mathrm{sign}(\delta)[\dots|\mathrm{diag}(b_{\delta(i')}, \overline{b_{\delta(i')}}^{-1}, I_{2(n-1)})|\dots|\mathrm{diag}(I_2, B_{\delta(j')})|\dots]$$

and for $1 \le i \le l$,

$$0 = \sum [a_1|\dots|a_i] \otimes [\mathrm{diag}(I_2, A_1)|\dots|\mathrm{diag}(I_2, A_{l-i})]$$
$$-\sum [b_1|\dots|b_i] \otimes [\mathrm{diag}(I_2, B_1)|\dots|\mathrm{diag}(I_2, B_{l-i})].$$

By the injectivity of $H_{l-i}(G_{n-2}) \to H_{l-i}(G_{n-1})$, we see that $y_2 = y_3$, (note that we view y_2 and y_3 as elements of T_2 or T_3). Now it is easy to see that $x = 0$. $\square$

Consider $R^{2(n-2)}$ as the submodule of R^{2n} generated by $e_5, e_6, \dots, e_{2n}$ (so G_{n-2} embeds in G_n as $\mathrm{diag}(I_2, I_2, G_{n-2})$). Let L'_* be the complex

$$\cdots \leftarrow C'_{n-3}(X_{n-2}) \leftarrow H_{n-3}(X_{n-2}, k) \leftarrow 0$$
$$0 \leftarrow 0 \leftarrow 0 \leftarrow k \leftarrow C'_0(X_{n-2}) \leftarrow C'_1(X_{n-2}) \leftarrow$$

with $X_{n-2} = \underline{\mathcal{IU}}(R^{2(n-2)})$. Define the map of complexes $L'_* \xrightarrow{\alpha_*} L_*$, given by

$$(\langle v_1\rangle, \dots, \langle v_k\rangle) \overset{\alpha_k}{\mapsto} (\langle e_1\rangle, \langle e_3\rangle, \langle v_1\rangle, \dots, \langle v_k\rangle) - (\langle e_1\rangle, \langle e_1 + e_3\rangle, \langle v_1\rangle, \dots, \langle v_k\rangle)$$
$$+(\langle e_3\rangle, \langle e_1 + e_3\rangle, \langle v_1\rangle, \dots, \langle v_k\rangle)$$

Note that this is similar to one defined in the proof of the proposition 2.6 in [8]. This gives the maps of bicomplexes

$$L'_* \otimes_{G_{n-2}} F'_* \to L_* \otimes_{G_n} F_* \to L_* \otimes_{G_n} F_*/L'_* \otimes_{G_{n-2}} F'_*,$$

where L_* and F_* are as in the proof of theorem 3.5 and F'_* is F_* as G_{n-2}-module, so it induces the maps of spectral sequences

$$E'^{r}_{p,q}(n) \to E^{r}_{p,q}(n) \to E''^{r}_{p,q}(n),$$

where all the three spectral sequences converge to zero. By a similar argument as in the proof of 3.5, one sees that the spectral sequence ${E'}^1_{p,q}(n)$ is of the form

$$ {E'}^1_{p,q}(n) = \begin{cases} E^1_{p-2,q}(n-2) & \text{if } p \geq 2 \\ 0 & \text{if } p = 0,1 \end{cases}. $$

For $2 \leq p \leq n$, ${E'}^1_{p,q}(n) \to E^1_{p,q}(n)$ is induced by inc : $R^{*p-2} \times G_{n-p} \to R^{*p} \times G_{n-p}$, $A \mapsto \mathrm{diag}(I_2, I_2, A)$, and

$$ {E''}^1_{p,q}(n) = E^1_{p,q}(n)/{E'}^1_{p,q}(n). $$

From the complexes

$$ \begin{array}{ll} D_*(q): & 0 \to E^1_{n,q}(n) \to E^1_{n-1,q}(n) \to \cdots \to E^1_{0,q}(n) \to 0 \\ {D'}_*(q): & 0 \to {E'}^1_{n,q}(n) \to {E'}^1_{n-1,q}(n) \to \cdots \to {E'}^1_{0,q}(n) \to 0 \\ {D''}_*(q): & 0 \to {E''}^1_{n,q}(n) \to {E''}^1_{n-1,q}(n) \to \cdots \to {E''}^1_{0,q}(n) \to 0 \end{array} $$

we obtain a short exact sequence

$$ 0 \to {D'}_*(q) \to D_*(q) \to {D''}_*(q) \to 0 $$

and by applying the homology long exact sequence to this short exact sequence we get the following exact sequence

$$ \begin{aligned} &{E'}^2_{n-1,q}(n) \to E^2_{n-1,q}(n) \to {E''}^2_{n-1,q}(n) \to {E'}^2_{n-2,q}(n) \\ &\qquad \to \cdots \to {E'}^2_{0,q}(n) \to E^2_{0,q}(n) \to {E''}^2_{0,q}(n) \to 0. \end{aligned} $$

Theorem 4.3. *Let $n \geq 1$, $l \geq 0$ be integer numbers. Then $H_l(\mathrm{inc}) : H_l(G_{n-1}) \to H_l(G_n)$ is surjective for $n-1 \geq l$ and is injective for $n-1 \geq l+1$.*

Proof. The proof is by induction on l. If $l = 0$ then everything is obvious. Assume the induction hypothesis, that is $H_i(G_{m-1}) \to H_i(G_m)$ is surjective if $m-1 \geq i$ and is bijective if $m-1 \geq i+1$, where $1 \leq i \leq l-1$. Let $n-1 \geq l$ and consider the spectral sequence ${E''}^2_{p,q}(n)$. To prove the surjectivity, it is sufficient to prove that ${E''}^2_{p,q}(n) = 0$ if $n \geq p+q$, $0 \leq q \leq l-1 \leq n-2$ and $2 \leq p \leq n-1$, because then we obtain ${E''}^2_{0,l}(n) = 0$ and applying lemma 4.1 we have the desired result. Let R^*_i denotes the i-th factor of R^{*m}. By the Künneth theorem ${E''}^1_{p,q}(n) = T_1 \oplus T_2 \oplus T_3 \mod {E'}^1_{p,q}$, where

$$ \begin{aligned} T_1 &= \bigoplus_{i_1 \geq 1} H_{i_1}(R^*_1) \otimes H_{i_3}(R^*_3) \otimes \cdots \otimes H_{i_p}(R^*_p) \otimes H_{q-\Sigma i_j}(G_{n-p}), \\ T_2 &= \bigoplus_{i_2 \geq 1} H_{i_2}(R^*_2) \otimes H_{i_3}(R^*_3) \otimes \cdots \otimes H_{i_p}(R^*_p) \otimes H_{q-\Sigma i_j}(G_{n-p}), \\ T_3 &= \bigoplus_{k_1,k_2 \geq 1} H_{k_1}(R^*_1) \otimes H_{k_2}(R^*_2) \otimes \cdots \otimes H_{k_p}(R^*_p) \otimes H_{q-\Sigma k_s}(G_{n-p}). \end{aligned} $$

Consider the following summand of ${E''}^1_{p+1,q}(n)$

$$U_1 = \bigoplus_{j_2, j_3 \geq 1} H_{j_2}(R_2^*) \otimes H_{j_3}(R_3^*) \otimes \cdots \otimes H_{j_{p+1}}(R_{p+1}^*) \otimes H_{q-\Sigma j_t}(G_{n-p-1}),$$

where $j_t = k_{t-1}$, $2 \leq t \leq p+1$. Let $\sigma_i^{(m)} := H_l(\alpha_{i,m})$. It is easy to see that $\sigma_1^{(p+1)}(U_1) \subseteq T_3$ and $\sum_{i=2}^{p+1}(-1)^{i+1}\sigma_i^{(p+1)}(U_1) \subseteq T_2$. Let $x = (x_1, x_2, x_3) \in \ker({d''}^1_{p,q})$. Since $n-p-1 \geq q-1 \geq q - \sum j_t$, by a similar argument as in the proof of lemma 4.2, we can assume that $x_3 = 0$. If

$$U_2 = \bigoplus_{j_2 \geq 1} H_{j_2}(R_2^*) \otimes H_{j_4}(R_4^*) \otimes \cdots \otimes H_{j_{p+1}}(R_{p+1}^*) \otimes H_{q-\Sigma j_t}(G_{n-p-1}),$$

then we have $\sigma_1^{(p+1)}(U_2) \subseteq T_1$, $\sigma_2^{(p+1)}(U_2) = 0 \mod {E'}^1_{p,q}(n)$ and $\sum_{i=3}^{p+1}(-1)^{i+1}\sigma_i^{(p+1)}(U_2) \subseteq T_2$. In the same way, using our assumption we can again assume that $x_1 = 0$. So $x = (0, x_2, 0)$. Once again we have $\sigma_1^{(p)}(T_2) \subseteq S_1$ and $\sum_{i=2}^{p}(-1)^{i+1}\sigma_i^{(p)}(T_2) \subseteq S_2$, where

$$S_1 = \bigoplus_{k_1 \geq 1} H_{k_1}(R_1^*) \otimes H_{k_2}(R_2^*) \otimes \cdots \otimes H_{k_{p-1}}(R_{p-1}^*) \otimes H_{q-\Sigma k_t}(G_{n-p+1}),$$

$$S_2 = \bigoplus_{l_2 \geq 1} H_{l_2}(R_2^*) \otimes H_{l_3}(R_3^*) \otimes \cdots \otimes H_{l_{p-1}}(R_{p-1}^*) \otimes H_{q-\Sigma l_t}(G_{n-p+1}).$$

By induction hypothesis $\sigma_1^{(p)}$ is an isomorphism, so $x_2 = 0$. Therefore ${E''}^2_{p,q}(n) = 0$ if $n \geq p+q$, $2 \leq p \leq n-1$, $1 \leq q \leq l-1$. To prove that ${E''}^2_{p,0}(n) = 0$ for $0 \leq p \leq n$, it is sufficient to prove that $E^2_{p,0}(n) = 0$ for $0 \leq p \leq n$. For $0 \leq p \leq n-1$ this follows from 3.5. If n is odd then $E^2_{n,0}(n) = 0$, because $d^1_{n,0}(n) = \mathrm{id}_k$. So let n be even. We prove by induction on n that $E^2_{n,0}(n) = 0$. If $n = 2$, then

$$\theta := (\langle e_1 \rangle, \langle e_3 \rangle) - (\langle e_1 \rangle, \langle e_1 + e_3 \rangle) + (\langle e_3 \rangle, \langle e_1 + e_3 \rangle) \in H_1(X_2, k)$$

and so $d^1_{3,0}(2)(\theta \mod G_2) = 1 \in \mathbb{Z}$. Assume that this is true for $n-2$, that is $E^2_{n-2,0}(n-2) = 0$. From the map ${E'}^1_{p,q}(n) \to E^1_{p,q}(n)$ we get the commutative diagram

$$\begin{array}{ccccc}
H_0(G_{n-2}, H_{n-3}(X_{n-2}, k)) & \xrightarrow{d^1_{n-1,0}(n-2)} & k & \longrightarrow & 0 \\
\Big\downarrow \alpha' & & \Big\downarrow \mathrm{id}_k & & \\
H_0(G_n, H_{n-1}(X_n, k)) & \xrightarrow{d^1_{n+1,0}(n)} & k & \longrightarrow & 0,
\end{array}$$

where the map α' is induced by the map α_*. By induction and the commutativity of the above diagram we see that $d^1_{n+1,0}(n)$ is surjective and therefore $E^2_{n,0}(n) = 0$. This shows that ${E''}^2_{p,0}(n) = 0$, $0 \leq p \leq n$ and so the proof of the claim is complete. To complete the proof of the theorem we must prove the injectivity claimed in the theorem. This can be done by a similar argument as in the above with suitable changes. $\square$

Corollary 4.4. *If $n-p \geq q$, then the complex*

$$\begin{aligned} H_q(R^{*p}\times G_{n-p}) &\xrightarrow{\tau_p} H_q(R^{*p-1}\times G_{n-p+1}) \xrightarrow{\tau_{p-1}} \cdots \\ &\xrightarrow{\tau_2} H_q(R^{*}\times G_{n-1}) \xrightarrow{\tau_1} H_q(G_n) \longrightarrow 0 \end{aligned}$$

is exact, where $\tau_i := d^1_{i,q}(n)$.

Proof. This comes out of the proof of 4.3. □

Theorem 4.5. *Let $n \geq 1$, $l \geq 0$ be integer numbers. Then $H_l(\mathrm{inc}) : H_l(G_n,\mathbb{Z}) \to H_l(G_{n+1},\mathbb{Z})$ is surjective for $n \geq l+1$ and is injective for $n \geq l+2$.*

Proof. For $n \geq l+1$, theorem 4.3 implies $H_{l+1}(G_{n+1},G_n) = 0$. Here $H_{l+1}(G_{n+1},G_n)$ is the homology of the mapping cone of the map of complexes $F_*^{(n)} \to F_*^{(n+1)}$ with coefficients in k where $F_*^{(m)}$ is the G_m-resolution of k. Applying the homology long exact sequence to the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

we have the exact sequence

$$\begin{aligned} \cdots \to H_{l+1}(G_{n+1},G_n,\mathbb{Q}/\mathbb{Z}) &\to H_l(G_{n+1},G_n,\mathbb{Z}) \\ &\to H_l(G_{n+1},G_n,\mathbb{Q}) \to H_l(G_{n+1},G_n,\mathbb{Q}/\mathbb{Z}) \to \cdots . \end{aligned}$$

We must prove that $H_{l+1}(G_{n+1},G_n,\mathbb{Q}/\mathbb{Z}) = 0$. Since $\mathbb{Q}/\mathbb{Z} = \oplus_p \varinjlim \mathbb{Z}/p^d\mathbb{Z}$ and since the homology functor commutes with the direct limit functor, it is sufficient to prove that $H_{l+1}(G_{n+1},G_n,\mathbb{Z}/p^d\mathbb{Z}) = 0$. This can be deduced from writing the homology long exact sequence of the short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^d\mathbb{Z} \to \mathbb{Z}/p^{d-1}\mathbb{Z} \to 0$$

and induction on d. Therefore $H_l(G_{n+1},G_n,\mathbb{Z}) = 0$. The surjectivity, claimed in the theorem, follows from the long exact sequence

$$\begin{aligned} \cdots \to H_{l+1}(G_{n+1},G_n,\mathbb{Z}) &\to H_l(G_n,\mathbb{Z}) \\ &\to H_l(G_{n+1},\mathbb{Z}) \to H_l(G_{n+1},G_n,\mathbb{Z}) \to \cdots . \end{aligned}$$

The proof of the other claim follows from a similar argument. □

Remark 3. Theorem 4.5 gives almost a positive answer to a question asked by Sah in [11, 4.9]. Also it gives better range of stability in comparison to other results [7], [15].

Let G be a topological group and let BG^{top} be the quotient space $\bigcup_n \Delta^n \times G^n/\sim$, where $\sim$ is the relation

$$(t_0,\dots,t_n,g_1,\dots,g_n) \sim \begin{cases} (t_0,\dots,\widehat{t_i},\dots,t_n,g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n) & \text{if } t_i = 0 \\ (t_0,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_n,g_1,\dots,\widehat{g_i},\dots,g_n) & \text{if } g_i = e. \end{cases}$$

It is easy to see that B is a functor from the the category of topological groups to the category of topological spaces. The topological space BG^{top} is called the classifying space of G with the underlying topology. Let BG be the classifying space of G as the topological group with the discrete topology. By the functorial property of B we have a natural map $\psi : BG \to BG^{\mathrm{top}}$.

Conjecture 4.6 (Friedlander-Milnor Conjecture)**.** *Let G be a Lie group. The canonical map $\psi : BG \to BG^{\mathrm{top}}$ induces isomorphism of homology and cohomology with any finite abelian coefficient group.*

See [6] and [11] for more information in this direction.

Theorem 4.7. *Let $F = \mathbb{R}$ or $\mathbb{C}$. If $G = O(F)$, $Sp(F)$ or $U(\mathbb{C})$, then $H_i(BG, A) \simeq H_i(BG^{\mathrm{top}}, A)$ for all i and any finite coefficient group A.*

Proof. See [4, Thm. 1, 2] □

Corollary 4.8. *Let $F = \mathbb{R}$ or $\mathbb{C}$. If $G_n = O_{2n}(F)$, $Sp_{2n}(F)$ or $U_{2n}(\mathbb{C})$, then $H_i(BG_n, A) \simeq H_i(BG_n^{\mathrm{top}}, A)$ if $n \geq i+1$ and any finite coefficient group A.*

Proof. This follows from 4.3 and 4.7. □

5. Homology stability of unitary groups over finite fields

In this section we will explain which part of the above results is true if R is a finite field, so in this section we assume that $R := F$ is a finite field.

Lemma 5.1. *Let F be a field different from $\mathbb{F}_2$. Then $\underline{\mathcal{U}}(F^n)$ is $(n-2)$-connected, $\underline{\mathcal{U}}(F^n)_w$ is $(n-|w|-2)$-connected for every $w \in \underline{\mathcal{U}}(F^n)$ and the poset $\underline{\mathcal{IU}}(F^{2n})$ is $(n-2)$-connected.*

Proof. The proof of the first two claims is by induction on n. Let $Z := \underline{\mathcal{U}}(F^n)$ and $Y := \mathcal{O}(\mathbb{P}^{n-2})$. For any $v = (\langle v_1\rangle, \dots, \langle v_k\rangle) \in Z\backslash Y$, there is an i, for example $i = 1$, such that $v_i \notin R^{n-1}$. This means that the n-th coordinate of v_1 is not zero. Choose $r_i \in F$ such that $v_i' = v_i - r_i v_1 \in F^{n-1}$, $2 \leq i \leq k$. It is not difficult to see that

$$\begin{aligned} Y \cap Z_v &\simeq Y \cap \underline{\mathcal{U}}(F^n)_{(\langle v_1\rangle, \langle v_2'\rangle, \dots, \langle v_k'\rangle)} \simeq Y \cap \underline{\mathcal{U}}(F^n)_{(\langle v_2'\rangle, \dots, \langle v_k'\rangle)} \\ &\simeq \underline{\mathcal{U}}(F^{n-1})_{(\langle v_2'\rangle, \dots, \langle v_k'\rangle)}. \end{aligned}$$

By induction $\underline{\mathcal{U}}(F^{n-1})_{(\langle v_2'\rangle, \dots, \langle v_k'\rangle)}$ is $((n-1)-(|v|-1)-2)$-connected, so $Y \cap Z_v$ is $((n-3)-|v|+1)$-connected. Since $Y \cap Z \subseteq Z_{(\langle e_n\rangle)}$, Z is $(n-2)$-connected [13, 2.13 (ii)]. To complete the proof we have to prove that $Z' := \underline{\mathcal{U}}(F^n)_w$ is $(n-|w|-2)$-connected. If $w \in Y$, then replacing Z by Z' in the above and using the induction assumption one sees that Z' is $(n-|w|-2)$-connected. If $w \notin Y$ then by induction $Y \cap Z'$ is $(n-|w|-2)$-connected and $Y \cap Z'_u$ is $(n-|w|-|u|-2)$-connected for every $u \in Z'\backslash Y$ as we proved in the above. Now by [13, 2.13 (i)] the poset Z' is $(n-|w|-2)$-connected. The proof of the last claim is similar to the proof given in Remark 1(ii). □

Lemma 5.2. *Let* $\mathrm{char}(F) \neq \mathrm{char}(k)$. *Then we have the isomorphism* $H_i(F^{*p} \times G_{n-p}) \simeq H_i(\mathrm{Stab}_{G_n}(\overline{\sigma_p}))$ *for all* i.

Proof. Let M be a finite dimensional F_1-vector space, where $F_1 := \{x \in F : \overline{x} = x\}$. From [1, Cor. 10.2, Chap. III] and the fact that for every group G, $H_i(G,k) \simeq \mathrm{Hom}(H^i(G,k),k)$, we deduce that $H_i(M,k) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$. By a proof similar to the proof of 3.4, one sees that $H_i(F^{*p} \times G_{n-p}, k) \simeq H_i(\mathrm{Stab}_{G_n}(\overline{\sigma_p}), k)$. □

Applying lemmas 5.1 and 5.2 one sees that theorem 3.5 is true if $F \neq \mathbb{F}_2$ and $\mathrm{char}(F) \neq \mathrm{char}(k)$. So we can apply the techniques that we developed in sections 3 and 4 to prove the following theorems.

Theorem 5.3. *Let* F *be a finite field different from* $\mathbb{F}_2$ *and* $\mathrm{char}(F) \neq \mathrm{char}(k)$. *Then*

(i) *the map* $H_l(\mathrm{inc}) : H_l(G_n) \to H_l(G_{n+1})$ *is surjective for* $n \geq l$ *and is injective for* $n \geq l+1$,

(ii) *if* $n - h \geq l$, *then the complex*

$$H_l(R^{*h} \times G_{n-h}) \xrightarrow{\tau_h} H_l(R^{*h-1} \times G_{n-h+1}) \xrightarrow{\tau_{h-1}} \cdots \xrightarrow{\tau_2} H_l(R^* \times G_{n-1}) \xrightarrow{\tau_1} H_l(G_n) \longrightarrow 0$$

is exact, where $\tau_i := d^1_{i,l}(n)$.

Theorem 5.4. *Let* $\mathrm{char}(F) = p$. *Then the map* $H_l(\mathrm{inc}) : H_l(G_n, \mathbb{Z}[\frac{1}{p}]) \to H_l(G_{n+1}, \mathbb{Z}[\frac{1}{p}])$ *is surjective if* $n \geq l+1$ *and is injective if* $n \geq l+2$.

Remark 4. Let F be a finite field such that $\mathrm{char}(F) = \mathrm{char}(k)$.

(i) We don't know if a similar results as 5.3 is true or not. There is some information from previous results, it is true if $n \geq 2l+3$ [7, Thm. 8.2].

(ii) Theorem 3.4 is not true in this case because otherwise it will be true with every prime field k as a coefficient group and so it must be true with integral coefficients (see proof of the theorem 4.5). Hence $R^{*p} \times G_{n-p}$ must be isomorphic to the group $\mathrm{Stab}_{G_n}(\overline{\sigma_p})$ [3], which is not true.

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